# UNITED STATES NAVAL POSTGRADUATE SCHOOL



## MUTUAL IMPEDANCE OF RHOMBIC ANTENNAS SPACED IN TANDEM

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JESSE GERALD CHANEY
PROFESSOR OF ELECTRONICS

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#### **ABSTRACT**

Upon examining the formulas for self and mutual impedances of antennas, it is found that while the mutual impedance formula for separately driven collinear standing wave antennas may be used directly in determining the radiation impedance when such antennas are connected in cascade, certain modifications must be made in the case of travelling wave antennas under similar circumstances. Accordingly, formulas are derived for rhombic antennas spaced in tandem and are modified to permit the determining of the radiation impedance of two identical rhombic antennas connected in cascade.

IN TANDEM VS. IN CASCADE

The mesh equations for two coupled antennas are

$$Z_{11}I_{1} + Z_{12}I_{2} = V_{1}$$

$$Z_{21}I_{1} + Z_{22}I_{2} = V_{2}$$

$$Z_{12} = Z_{21}$$
(1)

The radiation impedance parameters are given by certain double line integrals, in which both paths are along the same wire for a self impedance, whereas one path is along each wire for a mutual impedance.

In the case of an open wire full wave antenna fed at a current antinode, a conventional method for determining the driving point impedance is to set

$$V_2 = 0, \quad I_2 = -I_1$$
 (2)

and add the two equations, giving,

$$Z_{in} = 2(Z_{11} - Z_{12}) (3)$$

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<sup>1.</sup> J. G. Chaney, "A critical study of the circuit concept", J. Appl. Phys., 22, 12, 1429 (1951).



The validity of the above procedure rests in the fact that not only are the integration paths for equation (3) exactly the same as they would be if the paths were directly set up for the self impedance of the full wave antenna, but the assumed currents satisfy the restriction

$$I_1^*I_2 = Re(I_1^*I_2) = -|I_1|^2$$
 (4)

On the other hand, if the reference currents were not in time phase and equation (4) were not satisfied, say

$$I_2 = I_1 \exp(jkh) \tag{5}$$

the division by  $|I_1|^2$ , as required in the formulation of a self impedance from the expression for the complex power, instead of by  $I_1^*I_2$  as required for the formulation of the mutual impedance, would require the current factor,  $Re[f_1(P_1)^*f_2(P_2)]$ , to be replaced by the factor,

$$Re[f_1(P_1) * f_2.(P_2.)exp(jkh)]$$
.

In other words, a mutual impedance formula for two individually driven travelling wave antennas would require modification before it would be permissible to write the radiation impedance of two such identical antennas driven in cascade by the formula

$$Z_r = 2(Z_{11} + Z_{12}) (6)$$

#### RHOMBIC ANTENNAS IN TANDEM

It has been shown<sup>2</sup> that the mutual impedance of two terminated rhombic antennas is given by

$$jkZ_{12} = \int_{1} \int_{2} e(r_{12}) \left[ \frac{\partial^{2}}{\partial s_{1} \partial \bar{s}_{2}} - k^{2} \cos \theta(s_{1}, s_{2}) \right] g(ks_{1}, ks_{2}) ds_{1} ds_{2}$$
 (7)

in which

$$g(ks_1, ks_2) = Re[f_1(ks_1)^*f_2(ks_2)]$$

$$e(r_{12}) = r_{12}^{-1} exp(-jkr_{12})$$

$$f_1(ks_1) = current \ distribution \ function \ along \ antenna \ one$$

$$f_2(ks_2) = current \ distribution \ function \ along \ antenna \ two$$

$$\theta(s_1, s_2) = angle \ between \ directions \ along \ the \ two \ antennas$$

$$r_{12} = distance \ between \ points \ on \ the \ two \ antennas$$

$$k = 2\pi/\lambda$$

<sup>2.</sup> J. G. Chaney, 'Simplification for mutual impedance of certain antennas', U. S. Naval Postgrad. School Tech. Rpt. nd. 6, Nov., 1952.



For two identical, coaxial, and coplanar rhombic antennas, the same corresponding sides are parallel as in the case of the paths which were used for finding the self radiation impedance of a rhombic antenna<sup>3</sup>. Since the current distribution functions are the same as those for the two paths along the same antenna, it follows that the integrations in equation (7) need only to be carried out over the nonparallel wires<sup>3</sup>.

Let each leg of the rhombus be of length l, and let the vertex angles at the generator be  $2\alpha$ . Then, after postulating unattenuated travelling waves of current along each rhombic antenna, equation (7) becomes

$$Z_{12} = j120sin^{2} \alpha \left[ \int_{0}^{l} \int_{0}^{l} \cos k(x_{1} - x_{2}) e(r_{12}) dx_{1} dx_{2} + \int_{0}^{l} \int_{0}^{l} \cos k(x_{1} - x_{3}) e(r_{13}) dx_{1} dx_{3} - \int_{0}^{l} \int_{0}^{l} \cos k(x_{1} + x_{4}) e(r_{14}) dx_{1} dx_{4} - \int_{0}^{l} \int_{0}^{l} \cos k(x_{1} + x_{5}) e(r_{15}) dx_{1} dx_{5} \right]$$
(8)

or changing to the exponential form,

$$Z_{12} = -j60 \sin^2 \alpha [I_1 + I_2 + I_3 + I_4 - 2I_5 - 2I_7]$$
(9)

in which

$$I_{1} = \int_{0}^{l} \int_{0}^{l} exp \left[ -jk \left( x_{5} + x_{1} + r_{15} \right) \right] r_{15}^{-1} dx_{5} dx_{1} \tag{10}$$

$$I_2 = \int_0^l \int_0^l exp \left[ jk(x_5 + x_1 - r_{15}) \right] r_{15}^{-1} dx_5 dx_1$$
 (11)

$$I_{3} = \int_{0}^{l} \int_{0}^{l} exp \left[ -jk \left( x_{4} + x_{1} + r_{14} \right) \right] r_{14}^{-1} dx_{4} dx_{1}$$
 (12)

$$I_4. = \int_0^l \int_0^l \exp\left[ jk(x_4. + x_1 - r_{14}) \right] r_1^{-1} dx_4 dx_1$$
 (13)

$$I_5 = \int_0^l \int_0^l exp[jk(x_3 - x_1 - r_{13})] r_{13}^{-1} dx_3 dx_1$$
 (14)

$$I_7 = \int_0^l \int_0^l \exp\left[jk(x_2 - x_1 - r_{12})\right] r_{12}^{-1} dx_2 dx_1 \tag{15}$$

in which  $x_0$  is the distance between the driving points, and in which

$$r_{12} = \left[x_1^2 + x_2^2 + x_0^2 - 2x_0(x_1 - x_2)\cos\alpha - 2x_1x_2\cos2\alpha\right]^{\frac{1}{2}}$$
 (16)

$$r_{13} = \left[x_1^2 + x_3^2 + x_0^2 + 2x_0(x_1 - x_3)\cos\alpha - 2x_1x_3\cos2\alpha\right]^{\frac{1}{2}}$$
 (17)

$$r_{14} = \left[x_1^2 + x_5^2 + x_0^2 + 2x_0^2 (x_1 + x_4) \cos \alpha + 2x_1 x_4 \cos 2\alpha\right]^{\frac{1}{2}}$$
 (18)

$$r_{15} = \left[x_1^2 + x_5^2 + x_0^2 - 2x_0(x_1 + x_5)\cos\alpha + 2x_1x_5\cos2\alpha\right]^{\frac{1}{2}}$$
 (19)

<sup>3.</sup> J. G. Chaney, "Free space radiation impedance of a rhombic antenna", U. S. Naval Postgrad. School Tech. Ppt. no. 4, May, 1952.



Let

$$d_{1} = [x_{0}^{2} + (2l\sin \alpha)^{2}]^{\frac{1}{2}}$$

$$d_{2} = [x_{0}^{2} + l^{2} - 2lx_{0}c\cos \alpha]^{\frac{1}{2}}$$

$$d_{3} = [x_{0}^{2} + l^{2} + 2lx_{0}c\cos \alpha]^{\frac{1}{2}}$$

$$Z_{12} = R_{12} + jX_{12}.$$

and

After evaluating the definite integrals in equation (9), the following formula is obtained,

```
R_{1,2}/60 = \cos k(x_0 \sec \alpha) \{4Cik(x_0 \sec \alpha + l + d_3) + 4Cik(x_0 \sec \alpha + l - d_3) + 4Cik(x_0 \sec \alpha - l + d_2)\}
+4Cik(x_0 seca-l-d_2)-4Cik[(seca+1)x_0]-4Cik[x_0(seca-1)-2Cik(x_0 seca+d_1)
 -2Cik(x_0seca-d_1)-Cik[x_0(seca-1)+2l(1-cosa)]-Cik[x_0(seca-1)-2l(1-cosa)]
 -Cik[x_0(seca+1)+2l(1+cosa)]-Cik[x_0(seca+1)-2l(1+cosa)]
+sink(x_0 seca) \{4Sik(x_0 seca+l+d_3)+4Sik(x_0 seca+l-d_3)+4Sik(x_0 seca-l+d_2)\}
 +4Sik(x_0 seca-l-d_2)-4Sik[x_0(seca-1)]-4Sik[x_0(seca+1)]-2Sik(x_0 seca+d_1)
 -2Sik(x_0 seca-d_1) - Sik[x_0(seca+1) + 2l(1+cosa)] - Sik[x_0(seca+1) - 2l(1+cosa)]
 -Sik[x_0(seca-1)+2l(1-cosa)]-Sik[x_0(seca-1)-2l(1-cosa)]
-\cos k(x_0\cos a)\{2Cik(d_3+l+x_0\cos a)+2Cik(d_3-l-x_0\cos a)+2Cik(d_2+l-x_0\cos a)\}
                                                                                                       (20)
 +2Cik(d_2-l+x_2\cos\alpha)-4Cik[x_0(1+\cos\alpha)]-4Cik[x_0(1-\cos\alpha)]
+sink(x_0cosa)\{2Sik(d_3+l+x_0cosa)+2Sik(d_3-l-x_0cosa)+2Sik(d_2+l-x_0cosa)\}
 +2Sik(d_{2}-l+x_{2}\cos\alpha)-4Sik[x_{2}(1+\cos\alpha)]-4Sik[x_{0}(1-\cos\alpha)]
+\cos k(x_0\cos a+2l\sin^2a)\{Cik(d_1+x_0\cos a+2l\sin^2a)+Cik(d_1-x_0\cos a-2l\sin^2a)\}
 -2Cik(d_2+x_0\cos\alpha-l\cos2\alpha)-2Cik(d_2-x_0\cos\alpha+l\cos2\alpha)+Cik[(x_0-2\log\alpha)(1+\cos\alpha)]
 +Cik[(x_0-2l\cos\alpha)(1-\cos\alpha)]
+sink(x_0cosa+2lsin^2a)\{Sik(d_1+x_0cosa+2lsin^2a)-Sik(d_1-x_0cosa-2lsin^2a)\}
 -2Sik(d_2+x_0\cos a-l\cos 2a)+2Sik(d_2-x_0\cos a+l\cos 2a)+Sik[(x_0\cos a-l\cos 2a)(1+\cos a)]
--Sik[(x_0-2l\cos alk1-\cos a)]
+\cos k \left(x_0\cos \alpha - 2l\sin^2\alpha\right)\left\{Cik\left(d_1 + x_0\cos \alpha - 2l\sin^2\alpha\right) + Cik\left(d_1 - x_0\cos\alpha + 2l\sin^2\alpha\right)\right\}
 -2Cik(d_3+x_0\cos\alpha+l\cos2\alpha)-2Cik(d_3-x_0\cos\alpha-l\cos2\alpha)+Cik[(x_0+2l\cos\alpha)(1+\cos\alpha)]
 +Cik[(x_0+2l\cos a)(1-\cos a)]
+sink(x_0cosa-2lsin^2a)\{Sik(d_1+cosa-2lsin^2a)-2Sik(d_3+x_0cosa+lcos2a)\}
 -Sik(d_1 - x_0 \cos \alpha + 2l \sin^2 \alpha) + 2Sik(d_3 - x_0 \cos \alpha - l \cos 2\alpha) + Sik[(x_0 + 2l \cos \alpha)(1 + \cos \alpha)]
 -Sik[(x_1+2l\cos\alpha)(1-\cos\alpha)]
```



```
X_{12} = cos(x_0 seca) \{4Sik(x_0 seca+l-d_3)-4Sik(x_0 seca+l+d_3)-4Sik(x_0 seca+l-d_2)\}
 +4Sik(x_0seca-l-d_2)+4Sik[x_0(seca+l)]-4Sik[x_0(seca-l)]+2Sik(x_0seca+d_1)
 -2Sik(x_{0}seca-d_{1})+Sik[x_{0}(seca+1)-2l(1+cosa)]+Sik[x_{0}(seca+1)+2l(1+cosa)]
 -Sik[x_0(seca-1)-2l(1-cosa)]-Sik[x_0(seca-1)+2l(1-cosa)]
-sink(x_0seca)\{4Cik(x_0seca+l-d_3)-4Cik(x_0seca+l+d_3)-4Cik(x_0seca-l+d_2)\}
 +4Cik(x_0seca-l-d_2)+4Cik[x_0(seca-l)]-4Cik[x_0(seca-l)]+2Cik(x_0seca+d_1)
 -2Cik(x_0seca-d_1)+Cik[x_0(seca+1)-2l(1+cosa)]+Cik[x_0(seca+1)+2l(1+cosa)]
 -Cik[x_0(seca-1)-2l(1-cosa)]-Cik[x_0seca-1)+2l(1-cosa)]
+\cos(x_0\cos\alpha)\{2Sik(d_3-l-x_0\cos\alpha)+2Sik(d_3+l+x_0\cos\alpha)+2Sik(d_2-l+x_0\cos\alpha)\}
 +2Sik(d_2+l-x_0cosa)-4Sik[x_0(1+cosa)]-4Sik[x_0(1-cosa)]
+ sink(x_0 cosa) \{2Cik(d_3 - l - x_0 cosa) - 2Cik(d_3 + l + x_0 cosa) - 2Cik(d_1 - l + x_0 cosa)\}
 +2Cik(d_1+l-x_0\cos\alpha)+4Cik[x_0(1+\cos\alpha)]-4Cik[x_0(1-\cos\alpha)]
                                                                                                    (21)
-\cos k \left(x_0 \cos \alpha + 2 l \sin^2 \alpha\right) \left\{Sik\left(d_1 + x_0 \cos \alpha + 2 l \sin^2 \alpha\right) + Sik\left(d_1 - x_0 \cos \alpha - 2 l \sin^2 \alpha\right)\right\}
 -2Sik(d_2+x_0\cos\alpha-l\cos2\alpha)-2Sik(d_2-x_0\cos\alpha+l\cos2\alpha)+Sik[(x_0-2l\cos\alpha)(1+\cos\alpha)]
 +Sik[(x_0-2lcosa)(1-cosa)]
+sink(x_0cos\alpha+2lsin^2a)\{Cik(d_1+x_0cos\alpha+2lsin^2a)-Cik(d_1-x_0cosa-2lsin^2a)\}
 -2Cik(d_2+x_0\cos\alpha-l\cos2\alpha)+2Cik(d_2-x_0\cos\alpha+l\cos2\alpha)+Cik[(x_0-2\log\alpha)(1+\cos\alpha)]
 -Cik[(x_0-2l\cos\alpha)(1-\cos\alpha)]
-\cos k(x_0\cos\alpha-2l\sin^2\alpha)\{Sik(d_1+x_0\cos\alpha-2l\sin^2\alpha+Sik(d_1-x_0\cos\alpha+2l\sin^2\alpha)\}\}
 -2Sik(d_3+x_0cosa+lcos2a)-2Sik(d_3-x_0cosa-lcos2a)+Sik[(x_0+2lcosa)(1+cosa)]
 +Sik[(x_0+2l\cos)(1-\cos\alpha)]
+sink(x_0cosa-2lsin^2a)\{Cik(d_1+x_0cosa-2lsin^2a)-Cik(d_1-x_0cosa+2lsin^2a)\}
 -2 \operatorname{Cik} (d_3 + x_0 \cos \alpha + l \cos 2\alpha) + 2 \operatorname{Cik} (d_3 - x_0 \cos \alpha - l \cos 2\alpha) + \operatorname{Cik} [(x_0 + 2 l \cos \alpha)(1 + \cos \alpha)]
 -Cik[(x_0+2lcosa)(1-cosa)]
```

For two closely spaced identical rhombic antennas, equations (20) and (21) may be considerably simplified. At the frequency for which a rhombic antenna is given an optimum design, the leg length is almost always taken as an integral multiple of a half wave length with the parameter 2kl becoming an even multiple of  $\pi$ . Under these conditions, the formulas may be further simplified. However, when operating at a frequency different from the design frequency, 2kl is usually not an integral multiple of  $\pi$ , and the formulas must be more carefully examined in case it is desired to connect the antennas in cascade. Accordingly, select

$$x_0 = 2l\cos\alpha + b, \quad b \leq 2l\cos\alpha, \quad b \leq l \tag{22}$$

let

$$d_a = l(1+8\cos^2 \alpha)^{\frac{1}{2}}$$

and simplify.



```
R_{12}/60 = \cos 2kl \left\{ C + \ln \left( k \ln^2 \alpha \right) + 2 Ci 2kl - Ci k k \ln^2 \alpha \right\} - 4 Ci \left[ 2kl \left( 1 + \cos \alpha \right) \right] - 4 Ci k \left[ 2kl \left( 1 - \cos \alpha \right) \right]
 -Ci[4kl(1+cosa)]-Ci[4kl(1-cosa)]+4Cik(3l+d_4)+4Cik(3l-d_4)
+sin2kl\{2Si2kl-Si4kl-4Si[2kl(1+cosa)]-4Si[2kl(1-cosa)]-Si[4kl(1+cosa)]
 -Si[4kl(1-cosa)]+4Sik(3l+d_4)+4Sik(3l-d_4)
-2\cos(2kl\cos^2\alpha)\left\{Ci(2kl\sin^2\alpha)+Ci(2kl\cos^2\alpha)-2Ci[2kl\cos\alpha(1+\cos\alpha)]\right\}
 -2Ci[2klcosa(1-cosa)]+Cik(d_4+2lcos^2a)+Cik(d_4-2lcos^2a)
                                                                                                 (23)
+2sin(2klcos^2a)\{Si(2klsin^2a)+Si(2klcos^2a)-2Si[(2klcosa)(1+cosa)]\}
 -2Si[2klcosa(1-cosa)] + Sik(d_4+2lcos^2a) + Sik(d_4-2lcos^2a)
+\cos(2kl\cos 2a)\cdot \{Ci(4kl\cos^2a)+Ci(4kl\sin^2a)+Cik[4kl\cos a(1+\cos a)]\}
 +Cik[4klcosa(1-cosa)]-2Cik(d_4.+4lcos^2a-l)-2Cik(d_4-4lcos^2a+l)
+\sin(2kl\cos^2\alpha)\{Si(4kl\cos^2\alpha)-Si(4kl\sin^2\alpha)+Sik[4l\cos\alpha(1+\cos\alpha)]\}
 -Sik[4lcosa(1-cosa)]-2Sik(d_4+4lcos^2a-l)+2Sik(d_4-4lcos^2a+l)
X_{12} = \cos 2kl \{-2Si2kl + Si4kl + 4Si[2kl(1+\cos\alpha)] - 4Si[2kl(1-\cos\alpha)]
 +Si[4kl(1+cosa)]-Si[4kl(1-cosa)]-4Sik(3l+d_A)+4Sik(3l-d_A)
-\sin^2 k l \{C + \ln(k l \tan^2 \alpha) - 2Ci^2 k l + Ci^4 k l + 4Ci [2k l (1 + \cos \alpha)] - 4Ci k [2k l (1 - \cos \alpha)] \}
 +Ci[4kl(1+cosa)]-Ci[4kl(1-cosa)]-4Cik(3l+da.)+4Cik(3l-da.)
+2cos(2klcos2a){Sik(2lsin2a)+Si(2klcos2a)-2Si[2klcosa(1+cosa)]-2Si[2klcosa X
 (1-\cos\alpha)] +Sik(d_4+2l\cos^2\alpha)+Sik(d_4-2l\cos^2\alpha)}
                                                                                                 (24)
+2sin(2hlcos^2a)\{Ci(2hlsin^2a)-Ci(2klcos^2a)+2Ci[2klcosa(1+cosa)]\}
 -2Ci[2klcosa(1-cosa)]-Cik(d_a+2lcos^2a)+Cik(d_a-2lcos^2a)]
-\cos(2kl\cos 2\alpha)\{Si(4kl\sin^2\alpha)+Si(4kl\cos^2\alpha)+Si[4kl\cos\alpha(1+\cos\alpha)]\}
 +Si[4klcosa(1-cosa)]-2Sik(d_a+4lcos^2a-l)-2Sik(d_a-4lcos^2a+l)
+sin(2klcos2a)\{-Ci(4blsin^2a)+Ci(4klcos^2a)+Ci[4klcosa(1+cosa)]\}
 -Ci[4kl\cos\alpha(1-\cos\alpha)]-2Cik(d_{4}+4l\cos^{2}\alpha-l)+2Cik(d_{4}-4l\cos^{2}\alpha+l)
 in which C = 0.5772... is Euler's constant.
```

Upon selecting the leg length as an integral multiple of a half wave length, formulas (23) and (24) reduce to



```
7
```

```
R_{12}/60 = C + ln(klsin^2a) - Ci4kl + 2Ci2kl - 4Ci[2kl(1+cosa)] - 4Ci[2kl(1-cosa)]
   -Ci[4kl(1+cosa)]-Ci[4kl(1-cosa)]+4Cik(3l+d_4)+4Ci(3l-d_4)
-2cos(2klsin^2a\{Ci(2klsin^2a)+Ci(2klcos^2a)-2Ci[2klcosa(1+cosa)]\}
 -2Ci[2klcoxa(1-cosa)]+Cik(d_4+2lcox^2a)+Cik(d_4-2lcos^2a)]
-2sin(2klsin^2a)\{Sik(2klsin^2a)+Si(2klcos^2a)-2Si[2klcosa(1+cosa)]\}
 -2Si[2klcosa(1-cosa)]+Sik(d_4+2lcos^2a)+Sik(d_4-2lcos^2a)
                                                                                          (25)
-\cos(4kl\sin^2\alpha)\{-Ci(4kl\sin^2\alpha)-Ci(4kl\cos^2\alpha)-Ci[4kl\cos\alpha(1+\cos\alpha)]\}
 -Ci[4klcosa(1-cosa)]+2Cik(d_4+4\cdot lcos^2a-l)+2Cik(d_4-4\cdot lcos^2\cdot a+l)
-sin(4klsin^2a)\{-Si(4klsin^2a)+Si(4klcos^2a)+Si[4klcosa(1+cosa)]\}
 -Si[4klcosa(1-cosa)]+2Sik(d_4-4lcos^2a+l)-2Si(d_4+4lcos^2a-l)
X_{12}/60 = Si4kl - 2Si2kl + 4Si[2kl(1+cosa)] - 4Si[2kl(1-cosa)] + Si[4kl(1+cosa)]
 -Si[4kl(1-cosa)]+4Sik(3l-d_4)-4Sik(3l+d_4)
+2\cos(2kl\sin^2\alpha)\{Si(2kl\sin^2\alpha)+Si(2kl\cos^2\alpha)-2Si[2kl\cos\alpha(1+\cos\alpha)]\}
 -2Si[2klcosa(1-cosa)]+Sik(d_4+2lcos^2a)+Sik(d_4-2lcos^2a)
-2sin(2klsin^2a)\{Ci(2klsin^2a)-Ci(2klcos^2a)+2Ci[2klcosa(1+cosa)]\}
                                                                                         (26)
 -2Ci]2kl\cos\alpha(1-\cos\alpha)]-Cik(d_4+2l\cos^2\alpha)+Ci(d_4-2l\cos^2\alpha)\}
+\cos(4kl\sin^2\alpha)\{-Si(4kl\sin^2\alpha)-Si(4kl\cos^2\alpha)-Si[4kl\cos\alpha(1+\cos\alpha)]\}
 -Si[4klcosa(1-cosa)]+2Sik(d_4+4lcos^2a-l)+2Sik(d_4-4lcos^2a+l)
-\sin(4kl\sin^2\alpha)\{-Ci(4kl\sin^2\alpha)+Ci(4kl\cos^2\alpha)+Ci[4kl\cos\alpha(1+\cos\alpha)]\}
 -Ci[4klcosa(1-cosa)]+2Cik(d_4-4lcos^2a+l)-2Cik(d_4+4lcos^2a-l)
```

Formulas (20) and (21) are for any two rhombic antennas that are terminated, have equal leg lengths, are coplanar, and are coaxially spaced  $x_0$  between driving points. Formulas (23) and (24) are for any two identical rhombic antennas which are closely spaced in tandem. Formulas (25) and (26) are for two identical rhombic antennas whose leg lengths are an integral multiple of a half wave length and which are closely spaced in tandem. It will be shown that formulas (25) and (26) may also be used for rhombic antennas connected in cascade even though the leg lengths are arbitrary.

#### RHOMBIC ANTENNAS IN CASCADE

If instead of solving for the phase of  $I_2$  as in the case of separately driven antennas, the phase of  $I_2$  is to be specified with respect to  $I_1$  in accordance with equation (5), an examination of the integrations of equations (10) to (15), inclusive, reveals that formulas (20) and (21) require modification only in the sine and cosine coefficients of the sine integral and cosine integral functions within the braces. This modification consists



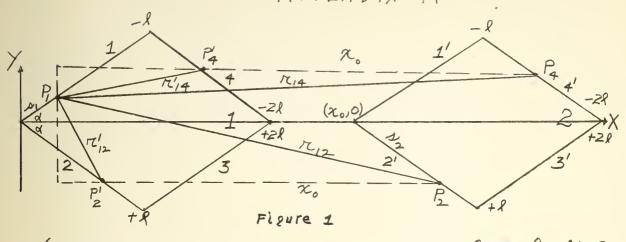
merely of adding the term kh to the argument of the sine and cosine coefficients. For example,  $cosk(x_0seca)$  is replaced by  $cosk(h+x_0seca)$ , etc. The algebraic sign of h is always the same as that of  $x_0$ .

If two identical rhombic antennas are to be connected in cascade, in order that a form of  $Z_{1\,2}$  suitable for substitution into equation (6) may be written, select h=-2l and modify equations (23) and (24). This again gives the formulas (25) and (26). Thus the particular form given for the mutual impedance of two closely spaced rhombic antennas in equations (25) and (26) is also valid for antennas connected in cascade even though the leg lengths are not an even multiple of a half wave length.

Hence, it follows that  $Z_{1,2}$ , as given by the latter two equations, may be used with the previously derived formula for the self radiation impedance of a single rhombic antenna  $^3$  for substitution into equation (6), giving the free space radiation impedance of two identical rhombic antennas in cascade.

Perhaps it should be emphasized that although the radiation impedance of a system of rhombic antennas determines the power radiated by the system, and is thereby useful in determining the gain of the system, it does not constitute the driving point impedance of the system. However, it does largely determine the attenuation of the current along the various legs of the system, and as a consequence does enter into the finding of the driving point impedance to a certain extent.





Variable s, is along rhombic 1, s is along rhombic 2  $-26 \text{ s} \le 0$ ,  $0 \le \text{s} \le +1$ Distance 12 is for paths (1,2')  $-21 \le \text{s} \le -1$ 14 is for paths (1,4')

now,  $R'_{12} = \sqrt{s_1^2 + s_2^2 + 2s_1 s_2 \cos 2d}$ Hence  $r = \sqrt{R'_{12}^2 + \chi_0^2 + 2 \chi_0 R'_{12} (s_2 + s_1) \cos d}$ 

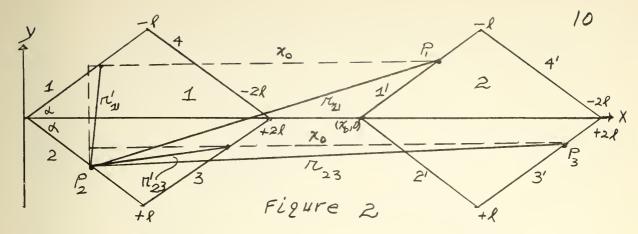
 $= \sqrt{2_1^2 + 2_2^2 + 2 \times_0 (4_1 + 2_2) \cos d + 2 \times_0 2 \cos 2d + 2 \times_0^2} \cos 2d + 2 \times_0^2 \cos 2d + 2 \times_$ 

Write 1 = 1 (1/2) - 2(1/2) (1/2+1) (1/2+1) co22 + (1/2+1)2

and  $r_{14} = \sqrt{(N_1 + R)^2 + (N_2 + R)^2 - 2(N_1 + R)(N_2 + R)\cos 2\lambda + x_0^2 - 2x_0(N_2 - N)\cos 2\lambda}$ 

Hence Ty = \( \langle (s,+l) + xcosd - (x+l)cos2d \rangle^2 + [x sind - (x\_2+l) sin2d \rangle^2





S, varies along rhombic 1, sparies along rhombic 2  $0 \le s, \le +l$ ,  $-l \le s \le 0$ Wistance  $R_2$ , is for paths (2, 1')  $R_2$  is for paths (2, 3')

 $R'_{21} = \sqrt{s_1^2 + s_2^2 + 2s_1s_2 \cos 2\lambda}$ 

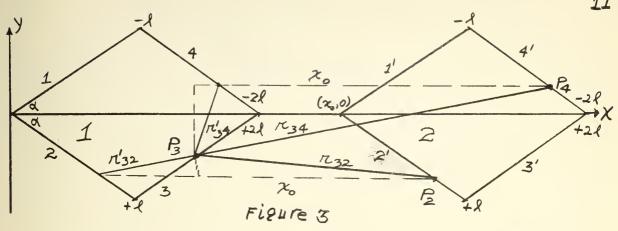
 $\pi_{21} = \sqrt{x_1^2 + x_2^2 + 2x_1 x_2} = 0.2 + \chi_0^2 - 2\chi(x_1 + x_2) \cos \lambda$ 

TC = \[s\_1 - x\_cos d + s\_2 cos 2d]^2 + [x\_sind-s\_2 sin 2d]^2

 $72_{23}^{\prime} = \sqrt{(l-s_1)^2 + (s_2-l)^2 + 2(l-s_1)(s_2-l)\cos 2\lambda}$   $= \sqrt{(s_1-l)^2 + (s_2-l)^2 - 2(s_1-l)(s_2-l)\cos 2\lambda}$ 

 $\Gamma_{23} = \sqrt{(x_1 - l)^2 + (x_2 - l)^2 - 2(x_1 - l)(x_2 - l)\cos 2d + 2\chi(x_2 - x_1)\cos d + \chi_0^2}$   $\Gamma_{23} = \sqrt{(x_1 - l) - \chi_0\cos d - (x_2 - l)\cos 2d}^2 + [\chi_0\sin d + (x_2 - l)\sin 2d]^2}$ 





s, varies along rhombic 1, s varies along rhombic 2 +R= N, = 28 -21 = B2 = - 8

Distance R32 is for paths (3,2') 1234 is for paths (3,4')

1232= /(1-12)2+(4-1)2+2(1-12)(N,-1) COS2X

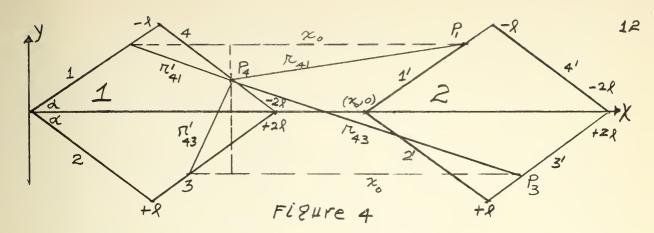
13= V[(s-1)-x, cord-(s2-1)cor2d]+[x, sind+(s2-1)sin2d]2

 $T'_{34} = \sqrt{(2l-N_1)^2 + (2l+N_2)^2 - 2(2l-N_1)(2l+N_2)} \cos 2d$ 

134=N(2R-21)2+(2R+2)2-2(2R-21)(2R+2)co2d+x2-2x(2+4)co2d

124= \[(0,-21)^2 x cost+(x+21) cos2d]2+[x sind-(2+21) sin 2d]2





S, varies along rhombie 1, so varies along rhombie 2  $-2l \le p_1 \le -l \qquad \qquad -l \le s_2 \le 0$   $+l \le s_2 \le +2l$ 

Vistance R<sub>41</sub> is for paths (4,1')

12 is for paths (4,3')

12/4 = N(R+21)2+(R+22)2-2(R+21)(R+22)Cos2x

12 = \((N+1)^2 + (N\_3+1)^2 - 2(R+0)(1+0)(002d + x^2 + 2x\_0(N\_1-N\_2) cord

12 = N[(s,+2)+x,cost-(sz+1)cos2d]2+[x,sina-(sz+1)cin2d]2

 $\pi'_{43} = \sqrt{(2l+n_1)^2 + (2l-n_2)^2 - 2(2l+n_1)(2l-n_2)\cos 2d}$ 

 $TC_{43} = \sqrt{(x_1 + 2x)^2 + (x_2 - 2x)^2 + 2(x_1 + 2x)(x_2 - 2x)\cos 2x + \chi^2 + 2\chi(x_1 + x_2)\cos x}$   $TC_{43} = \sqrt{(x_1 + 2x) + \chi\cos x + (x_2 - 2x)\cos 2x} + \sqrt{2x_1 + 2x}\cos x + (x_2 - 2x)\cos x + \sqrt{2x_2 + 2x}\cos x + \sqrt{2x_2 + 2$ 

 $\Gamma_{41}(N_1, N_2) = \Gamma_{14}(N_1, N_2)$   $\Gamma_{21}(N_1, N_2) = \Gamma_{12}(-N_1, -N_2)$ 

132(1,12)=123(1,12) 1243(1,12)=134(-1,-12)



Hiven  $jKZ_{12}/30 = f_1f_2e(r_{12})(\frac{3^2}{3s_23s_1} - K^2\cos\theta_{12})g(ks_1, Ks_2)ds_1ds_2$  (1)

with  $g(Ks_1, Ks_2) = ReIf_1(R)^*f_2(R)$ ,  $\cos\theta_{12}ds_2 = dr_1 \cdot dr_2$ .

Ofter postulating travelling waves a current along each rhombic antenna and considering the directions of the various legs of each antenna,

 $\frac{Z_{12}}{J_{12}} | J_{13} | K_{0} \sin 2 J_{0} | = \int_{0}^{1} \int_{0}^{1} \frac{C_{02} K(a_{1} + a_{2}) e(n_{1}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} + a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} + a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} + a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} + a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} + a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} + a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2} da_{1}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}{c_{12} - 2 \int_{0}^{1} \frac{C_{02} K(a_{1} - a_{2}) e(n_{2}) da_{2}}$ 

Since  $R_{21}(N_1,N_2)=R_2(-N_1,-N_2)$ ,  $R_2(N_1,N_2)=R_2(-N_1,-N_2)$  $R_{41}(N_1,N_2)=R_{14}(N_1,N_2)$ ,  $R_2(N_1,N_2)=R_2(N_1,N_2)$ 



12,4=NE(0+1)+x,cox-(0,+1)cox2d]2+[x,sind-(x+1)sin2d]2 123=N[(x,-2)-x, cord-(2-2)cor2d]+[x,sind+(2-2)sin2d]2 12 = N[(2,-21)-x cood+(2+21)co22d] = [xsind-(2+21)sin2d]2 17=1(s1+x0cos++s2co=2d)2+(xosind+s2sin2d)2 equation (2) becomes, Z/5/20Ksin2d=50 5 & cosk(s,+s2) e(12) ds2 ds,  $+ \int_{s=1}^{2l} \int_{s=-2l}^{-l} \cos k(s+s_2) e(r_{34}) ds_2 ds_1$ (3) - (S=+ S=-2 + S=-2 S== ) Co2 K(s\_1-s\_2) e(r\_4) ds\_2 ds,  $-\left(S_{1=0}^{2} \int_{y=2}^{2l} + S_{1=2}^{2l} \int_{y=0}^{2}\right) \operatorname{Coak}(s_{1} - s_{2}) e(r_{3}) ds_{2} ds_{4}$ In the first integral of equation (3), let s = -x,  $S_2 = \chi_2$ ;  $R_1 = \sqrt{(\chi_1 + \chi_2 \cos 4 - \chi_2 \cos 2d)^2 + (\chi_2 \sin d + \chi_3 \sin 2d)^2}$ In the second, let  $x_1 = -\chi_1 + 2l$ ,  $x_2 = \chi_3 - 2l$ ; 134 N= N(X+X0-X3co2d)2+(Xosin +-X3sin2d)2 In the first of the third, let x=x,-2, 3=-2,-2; 17= V(X,+ X, cosd+ X, cosed)2+ (x, sind+ x, sin 2d)2 In the second of the third, let a =- x-l, s= x-l;  $14 NR = \sqrt{(x-x\cos x+x\cos 2x)^2+(x\sin x-x\sin 2x)^2}$ In the first of the fourth, lets, =- x, +l, == x+l; 123 ~ N14 9 m the second of the fourth, let s = x+l, 2=-x+l; 123 ~ MIS-Equation (3) then becomes,



Zn fix/20 sind = So Scark(x,-x)e(r,2)dx,dx+ + Scork(x,-x3)e(r,3)dx,dx3 - So Scork(x+x) (1,4) dxdx- S Book(x+x) e(75) dxdx5 Changing to the exponential form, (5)

Z\_=-jk60ain2 [I,+I2+I+I-I-I-I-I8]

with

I= 5 \$ 5 l exp[-ix(12+x,+12)] 12=1 dx dx, I= SIS exp[]'K(x5+x-25)]2= dx5dx, I= 5252 exp[-jk(x4+x,+n,4)] 12-1 dx dx, I = SlSlexp[sk(x+2,-1,4)] 12/4 dzdx, I= Sistexp[]:K(x3-x-13)] 12-1 dx,dx, I= 5) Stexp[-31(x3-7,+1,3)] 12-1 dx3 dx, I= SISPERP[-1K(2-7-12)] 11-1 dx2dx, I= Sistemp[-sk(2-x+1,2)] 12 dx dx, 12= 12+x2+x2-2x0(x-x1)cod-2x, xco2x 13=17,2+23+x2+27,(x-73) cord-IX, x3cor2+ 12=N x2+ x2+ x2+2x(x+x) cosd+2x, xcos2x 15=1 x2+ x3+ x2-2x (x,+x) GO d+27, x GO 2d In Ig, let 2 - x, , x, -x; in I, let 2 -x, x, + 2; then

Z=-160ksin2 [I,+ [+ I+ I+ I-2 I-2 I]



In carrying out the integrations, the following limits are encountered;



Given  $I_{,=} \int_{0}^{2} \int_{0}^{2} e^{-jk} (x_{5} + x_{i} + x_{i} =) dx_{5} dx_{i}$  (1)

7,5= V(x-xcod+xco22d)2+(xsind-xsin2d)2 = 1/2+2+2+x2-27(x+x)co2d+2x, xco22d

Let  $m_1t = x_1 - x_1\cos 2x + x_2\cos 2x + x_1 = m_1 = x_1\sin 2x = m_1t_1 = -x_2\cos 2x + x_2\cos 2x + x_2 = x_2\cos 2x + x_2\cos 2x +$ 

 $m_1 dt = (1 + \frac{x_1 - x_6 \cos d + x_5 \cos 2d}{\pi_{15}}) dx_1 = \frac{m_1 t dx_1}{\pi_{15}}, dx_2 = \frac{n_1 t dx_1}{\pi_{15}}$ 

 $\frac{\partial (m_1 t)}{\partial x_5} = \cos 2d + \frac{x_5 - x_0 \cos d + x_1 \cos 2d}{\pi_{15}} = \frac{x_5 - x_0 \cos d + (x_1 + x_2) \cos 2d}{\pi_{15}}$ 

 $\frac{1}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_5} = \frac{x_5 - x_0\cos\alpha + x_$ 

 $\frac{1}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_5} = \frac{\chi_5 - \chi_0\cos\alpha + (l + R_{25})\cos2\alpha}{R_{25}(l - \chi_0\cos\alpha + \chi_0\cos2\alpha + R_{25})}$ 

 $X_5 + x_1 + N_5 = m_1 t + x_0 \cos \alpha + 2 x_5 \sin^2 \alpha$   $I = e^{i \kappa x_0 \cos \alpha} \int_0^{R} e^{i \kappa x_2} x_5 \sin^2 \alpha \int_{\kappa m_1 t_1}^{\kappa m_1 t_2} e^{-i t} dt \qquad (2)$ 



I = = -inx cod [= laxx sind (cirmt-cirmt,)-i(sirmt-sirmt,)] + = ikx cood pl-jexx sind = ikm to (m,to) = ikm, to 2 (m,to) } d? Let -ikxcood Pizkx sind ( Simila ) (mit) } dxs =  $\int_{0}^{1} e^{i\kappa(n_{RS}+l+x_{S})} [x_{S}-x_{s}\cos d+(l+n_{RS})\cos 2d] dx_{S}$ - Sheik(x+ros)[x=xcood+roscoodd] dx= A,=- [ =- [ =- ik(x,+10,5) [x,-x cood+12,5cood] dx, [x= kx-xcod] = (6) Let my=12+(x5-xcood), m2 4 = 120-20 cod = x0(1-cod) = m, t, ]. m24=102+1-xcoa=m12]0, x5+10=m24+x0coad m2 dy = [1+ x5-x50+ ]dx = m2 y dx5, dx = 25 dy m2 = (xosind), = 105 - x5 + x0 CDd 105= m3 (43+1) 25= m2 y+2 cost-20= m3 (y2)+26 cost 95= - m2/y+ x, cod+ 10,5

xocod+x5-x5co2d+l-75cod+x5co2d+x5=l+x+x5 x cod + x3 - 75 - Co2d + x5 co2d + 120 = 25 + 1205



 $A_{12} = -e^{-jKx_{0}\cos2d/\frac{y_{1}}{y_{1}}} e^{-jKm_{2}y_{1}} \left[ \frac{m_{2}(y^{2}-1) + m_{2}(y^{2}+1)\cos2d}{\frac{1}{2}y_{1}} (y^{2}+1)\cos2d + \frac{m_{2}(y^{2}+1)}{\frac{1}{2}y_{1}} (y^{2}+1) \right]$ =- eikxcord (3/2 e-ikmy (y cos d-sind) (y cosd+sind) dy =- e'Kx, cost / y2 e-skm24 (y+tand) dy

A12 = - e'Kx, cost / y2 -ikm24 (2 tand) dy

Y (y-tand) dy (7)

Let  $-u=km_2(y-tand)=km_2y-kx_0\frac{\sin^2d}{\cos d}$ -JKm2y = J'22-J'KK Singst 2,=-K(m2y,- xo sind tand) = - K8(1-cod- sin2d) = KX ( sec d - 1) 21=-K(my-xsindtand)=-K[rotl-x(cost+sind)) = Kxosecd - (rox+l)}

Ietu = Km2y, n,=Km2y,=Kx(1-co2d) 2= Km2 y2 = K(20+1-x0000)

A1=-Zeikxaed {cik[xseed-(rat)]-cik[xo(seed-1)] +J[SiK[xosecd-(Pos+1)]-SiK[xo(secd-1)]} + eik&cood { Cix[roth-xand]-cix[x (1-cood)] }
- J'[Six[roth-xand]-Six[xo (1-cood)] }



```
A_{13} = \int_{0}^{1} e^{-\frac{i}{\lambda} \left( \frac{1}{25} + 1 + \frac{1}{\lambda} \right) \left[ \frac{1}{25} - \frac{1}{25} \cos \alpha + (1 + \frac{1}{25}) \cos \alpha} \right]} dx_{3} - \frac{1}{125} \left[ \frac{1}{25} - \frac{1}{25} \cos \alpha + \frac{1}{25} \cos \alpha} \right] dx_{3} - \frac{1}{25} \left[ \frac{1}{25} - \frac{1}{25} \cos \alpha + \frac{1}{25} \cos \alpha} \right] dx_{3} - \frac{1}{25} \cos \alpha} dx_{3} -
                                                                                                                                                                                                                                                   (9)
                                   125= 1(x5-xcod+8cos2d)2+(xsind-8sin2d)2
                                                                                                                                                                      m = x sind-lain 2d
       M2 y= 18+25-2000 d+lco22d
       M2 4 = 120+1co224-76 co2 a
                                                                                                                                                                  120=1702+12-2× 2002d
         my= rep +28costd-x, cosd
                                                                                                                                                              120=122+(2/co2d)2-4x/co2d
          m dy = [1+ x5-x6 cood+ 1 cosad ] dx = m y dx, dx = 125 dy
             75+75+8= m2y+x0co2d+21 sined
           m2 = (x, sind-lained)2

13 = 125+[x,-(x, cond-losed)]
                                                                                                                                          = 125-[x,-(x,cood-lcood)]
                                                                                                                                         (xcod-fcoad)coad =
               12 = m2 (32+1)
                                                                                                                                         1 % cod-2 % cod sin 2 - 8 coz 2 d =
                  25= mzy-125+x cood-leased
                                                                                                                                         12-x cood + (x cozd-2coz 2d) coz 2d =
                                                                                                                                        I I sin 2 2 d - 7 sind sin 2 d
                    2=-m2+12+xcood-scor2d | & sin220
= m2 (y2-1)+(xcood-scor2d)
             A_3 = eik(xcosa+2 lained) ( = ikm y m2 (y2-1)+ m2 (y2-1) cos2d d 2 (y2-1)+ m2 (y2-1) cos2d-m2 sin2d
                          = e 1 k(x cosd+2 lain a) (= 1 km 4 y 2 cos 2 d - sin 2 dy y 2 cos 2 d - 2y sind cosd + sin 2 d dy
                             = ei(xcod+2losnid)( =-ikmoy( = tand - 1) dy
                                                                                                                                                                                                                                                           (10)
              Let
                   2=- Km2 (y-tond), - Km2 y= 4- Km2 tond = 4- Kx sind + 2 K/ sind
                    2=-K(12+22-2 seed), 21=-K(120+2-2 seed)
                   Let = Km24
                             u= K(ng+2/costd-xocosd)
                               24 = K ( 120 + 8 cos 2d - 70 cos d)
```



A13 = 2 e - ik x accor (cik / 12 + 2 R - x seed | - cik / 12 + 2 - 2 seed | -j[Sik(120+21-20sea)-Sik(120+1-20sea)]} - eik(x,coa+2/sind){Cik(12+2/co3d-xcod)-Cik(12+2/co2d-xcoad) -j[six(rge+22cost-2cosd)-six(rg+1cos2d-2cosd)]} Substituting into equation (3), -jI,2Ksind = 2 e-1'K % seed { Cir(x, seed-l-rox) - Cir[x, (seed-1)] - Cik/130+21-7, seed + Cik/130+1-2, seed | + J'[Sir(xseed-l-102)-Sir[xoleeed-1)] + Sir(122+21-xseed) -Sik(rgo+1-2, sec a)]} (12)-2 = 1 Kx coascik(12+1-x coad) - cik[x (1-coad)] -1'[six(rg+1-xcood)-six[x,(1-cood)]} +2 = 1 K(x cos d+2 l sixtd) { CiK(12+2 l cos2d-7, cos d) - Cik(120+2cm2d-xcmd)-j[5ik(12+2lcm2d-xcm2d) -Sir (1/20+1 cus 2 x - 7, cos x)]} upon breaking equation (12) into its real and imaginary components,



-sksin2d I, = Cos (Kx seed) {2cik(x seed-l-1/x2+l2-2lxcod)-cix[x(seed-1)] -Cil(Xo(secd-1)-28(1-cood)]} + Sin (Kx seed) {2Sik(x seed-l-1/k2+l2-2/x cod))-Sik[x (seed-1)] -Sik[x (seed-1)-2 (1-cod)]} - Co(KXCod){CiK(VX2+12=21XCod+1-XCod)-CiK[X(1-cod)]} toin(Kxcozd)[Sik(Vz2+8=28xcozd+8-xcozd)-sik[x(1-cozd)]] + Cos K(2 cosd + 2 l sind) {CiK[[x=2 \cusd]-cosd]-cosd]-- Cik (1/x2+12-2/x cos x+lous 2d-x cosd)} - sink(2 cood+2/sind) {six[(2-2/cood)(1-cood) - Sik(Vx2+8221xcosd+1cos2d-xcosd)} +i (2 seed) {25ik(xseed-l-1/x2+l22lxcood)-5ik[x(seed-1)] - SiK[76 (seed-1)-21 (1-Cood)]} - sin K(xseed) {2CiK(xseed-l-1/x2+2221xcoot)-CiK(x(seed-1))] - Cik[x (seed-1)-28(1-cod)]} + cos x(x cosd){six(/x3+8=21x cod+1-xcosd)-six[x (1-cod)]} +sinK(xcosd){Cik(\z2+1=21xcosd+1-xcosd)-cik[x(1-cosd)]} - Cosk(x, cosd + 2/sin2d) {5ik[(x,-2/cosd)(1-cosd)] -Sik(Vx2+12-21x, cos & tlcos2d-x cos d)} -  $sin k(x cos d + 2 l sin^2 d) \{ cik[(x - 2 l cos d)(l - cos d) \}$   $- cik(\sqrt{x^2 + l^2 - 2 l x cos d} + l cos 2 d - x cos d) \}$ 



Given 
$$I_2 = \int_0^1 \int_0^1 \frac{e^{-jk(R_j - x_j - x_j)}}{R_j + x_j} dx_j$$
 (1)

 $I_1 = \sqrt{(x_j - x_j)} \frac{e^{-jk(R_j - x_j - x_j)}}{R_j + x_j} \frac{e^{-jk}}{R_j +$ 



Let ikxcood of ikax sind = ikm, to d(m,t) = ikm, to d(m,t) ]dx

Azi = e se se ikax sind = ikm, to d(m,t) = ikm, to d(m,t) ]dx

xcood+2xsin2d-R=xcood+xsco2d=xs-ros x, cosd+2x, sin2d-1,+8-x, cood+x, coo2d=2+1-125

 $A_{21} = \int_{0}^{2} e^{jk(l+\chi_{5}-r_{2}s)[\chi_{5}-\chi_{c}cod+(l-r_{4}s)cos2d]} d\chi_{5}$   $\frac{\pi_{25} \sum_{R_{5}} r_{4}-l+\chi_{c}cos2d-\chi_{5}cos2d}{r_{2}cos2d}$ - Sleik(x-105) [x-x, cood-105 coold] dx105[105+x, cood-125 coold] (5)

Set  $A_{22} = -\int_{0}^{1} \frac{e^{iK(x_{5}-N_{0.5})} \left[\chi_{5}-\chi_{0} \cos d-N_{0.5} \cos 2d\right]}{N_{0.5} \left[N_{0.5}+\chi_{0} \cos d-N_{0.5} \cos 2d\right]} dR_{5}$  $R_{05} = \sqrt{(\chi_5 - \chi_0 \cos \alpha)^2 + (\chi_0 \sin \alpha)^2}$ 

 $m_2y = R - (\chi - \chi_0 \cos \lambda)$ ,  $m_2 = \chi_0 \sin \alpha$ 

m24,=120+x0cord=x0(1+cord)=m,t,]0

m2 y2= 10, -1+2, cod = m, t2 Jo

 $m_2 dy = \left[ \frac{\chi_5 - \chi_0 \cos \alpha}{R_0 5} - 1 \right] d\chi_5 = -\frac{m_2 y}{R_0 5} d\chi_5, d\chi_5 = -\frac{R_0 5}{y} dy$   $\chi_5 - R_0 = -m_2 y + \chi_0 \cos \alpha \qquad \chi_0 \cos \alpha (1 - \cos 2\alpha) = 2\chi_0 \cos \alpha \sin^2 \alpha$ 

 $\frac{M_2}{y} = \frac{(\pi \sin \alpha)^2}{\pi_5 + \chi \cos \alpha} = \pi_5 + \chi_5 - \chi_6 \cos \alpha$   $= \chi_5 \sin \alpha \sin \alpha \cos \alpha = \pi_5 + \chi_5 - \chi_6 \cos \alpha$ 

15 = m2 (y2+1), x-xcosd = - m2 (y2-1)

 $A = e^{\frac{1}{3}kx\cos 2} \frac{y^2}{y^2} \frac{-\frac{1}{3}km_2y}{y^2} \frac{m_2(y^2-1) - \frac{m_2}{2y}(y^2+1)\cos 2x}{y^2} \frac{1}{y^2} \frac{m_2(y^2-1) - \frac{m_2}{2y}(y^2+1)\cos 2x}{y^2}$ 

= - e'kxcod (32 - 1'km 4 (2 + tand - 4) dy (8)

Let u= Km (y+ tand) = Km y+ Kx sin2d ,- jkm y=-ju+jkx sin2d cosa  $u = K(n_{oq} - l + \chi_{oseed})$ ,  $u_i = K(\kappa_o(seed + 1))$ 

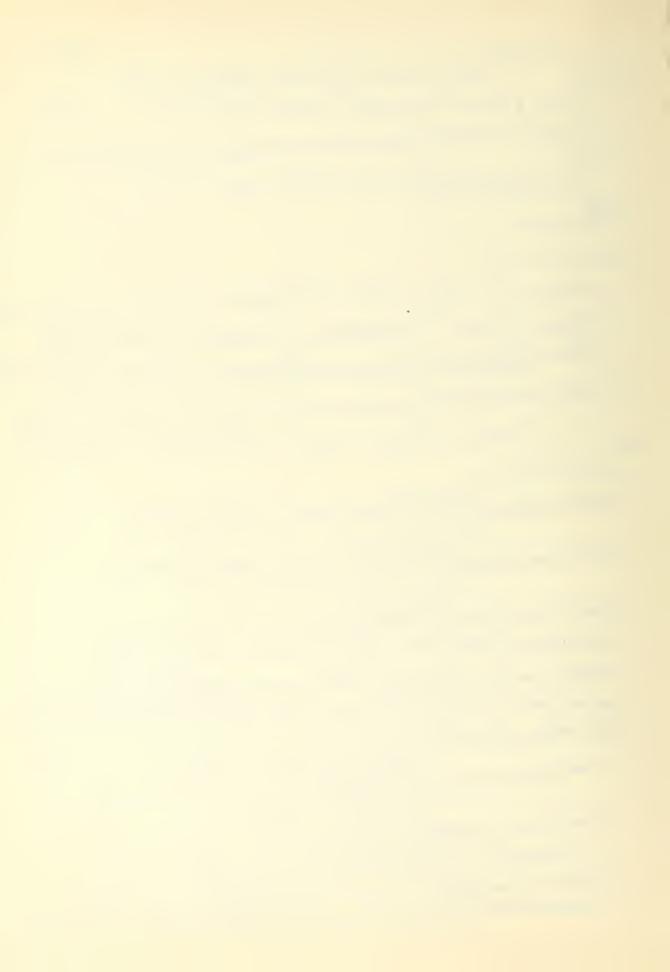
~= Km2y, ~= K(10/2-1+x cod), ~= KX(1+ cod)



A2=-2e {ak(r= l+xsecx) - Cik[xo(secd+1)]}
-J'[Sik(r= l+xsecx) - Sik[xo(secd+1)]} (9)+ expand(cik(rol-1+xcood)-cik[xo(1+cood)] -1/51x(ng-l+xcord)-Six[xo(1+cord)]} Set  $A_{23} = \int_{0}^{1} e^{jk(l+x_{5}-R_{85})} \frac{1}{2x_{5}-x_{5} \cos d+(l-R_{85}) \cos 2d} dx_{5}$ 125 = V(x-7, cosd+1 cos2d)2+(x, sind-1 sin2d)2 my= 125-x+xcusd-lcos2d, m= x sind-lsin2d my= 70(1+cosa)-l(1+cosad)-2lcosd, 12=122+12=2lxcosd = (x-2/cosa)(1+cosa)= m,t,], The= x-2/cosa m27,= 120+xces d-lcos 2d = m,t,] m2 dy = [x5-xcoa+leased-1]dx = - m27 dx, dx = - 725 dy  $l+\chi-R_{5}=-m_{2}y+\chi \cos d+2l \sin^{2}d$  $\frac{m_2}{y} = \frac{(\chi \sin \alpha - l \sin 2\lambda)^2}{\pi_{g_5} - (\chi - \chi \cos \alpha + l \cos 2\lambda)}$ = 125 x - x cord+lcor 2d 12 cosd(1-cos2d)-P(1-cos22d) 1 = m2 (y2+1)  $\chi_{5} \chi_{5} cord + l cord = \frac{m_{2}(y^{2}-1)}{2y} - \frac{m_{2}(y^{2}-1)}{2y} cord + 2 l sind) \int_{y_{2}}^{y_{2}-1} km_{2}y - \frac{m_{2}(y^{2}+1)}{2y} cord + \frac{m_{2}(y^{2}-1)}{2y} (y^{2}-1) dy$   $y_{1} \frac{y_{1} m_{2}(y^{2}+1) + m_{2}(y^{2}-1) cord + m_{2} sind}{2y}$   $- \frac{1}{2} \frac{1}{2}$ =2xcordsing-psinza =sinza(xsina-lsinza)=msinza = e sk(2 cord+2 laind) [ 3/2 - 1 Km2 y [ 2 y tand - 4] dy (11) u= Km2(y+tand)= K(m2y+ x sin2d-2 (sin2d), - Kmy=-Ku+K(x sin2d-2 (sin2d) ~= K[xo(seca+1)-28(1+cod)], ~= K[rgo+ x seca-8] u = - Km2y, u= K(x-2/cost)(1+cost)=m,t] = K(7+xcost-fcost)=m,t]



A\_3=2e {Cik[x(seed+1)-2l(1+cood)]-Cik(/z+xseed-l)}
-j[Sik[x(seed+1)-2l(1+cood)]-Sik(/z+xseed-l)]} (12)- E'K(x coad+2lsin2d) scikl(x=2lcoad)(1+coad)]-Cik(12+xcoad-2coa2d) -j[Six[x-2/cord)(1+cord)]-Six(120+xsecd-2)]} Therefore -j2Ksin2d I= es'kx secd {2cik[no+ x secd-l]-cik[x (secd+1)]-cik[x (secd+1)-2l(4cod)]
-1[28ix[no+xsecd-l]-sik[x (secd+1)]-sik[x (secd+1)-2l(1+cod)]} -eitzcora { ix[rotxcord-s]-cix[x(1+cord)]-j[ix[rotxcord-s]-six[x(1+cord)]]} (13) + esk(x, cord+28sind) {cik[(x-21cord)(1+cord)]-cik(1/20+xcord-1corrd) or -j[Sik[(x-2lcod)(1+cod)]-Sik(ryo+xcod-lcoz2d)]-j $Ksin^2dI_2=$ Cos(tx seed) [2cik(Vx2+x2=21xcood+xsecd-x)-cikix seed+v] -CiK[x, (secd+1)-2l(1+cosd)]} + sin(Kx sect){25ik(x2+12-2lxcod+x secd-1)-sik[x(secd+1)] - Sik[r(sea+1)-28(1+cod)]} -Go (Kx cood) {Cik(\x2+1=2/xcood+xcood-1)-cik[x0(1+cood)]} -Din(Kx cood) {S(k(\x2+1=2/xcood+xcood-1)-sik[x0(1+cood)]} + Cosk(x cood+2/sind){Cikpx-2/cood(1+cood)]-Cik(x2+x22/xcood+xcood-2co2d)}
+ sink(xcood+2/sind){Sik[x-2/cood)]-Sik(x2+x22/xcood+xcood-2co2d)} + 1 [- Coskx seca) {25 in(1/2+122/x Cood+x seed-1)-5 in[x (seed+1)]
-Sin[x (seed+1)-2! (1+cood)]}
+ nin (xxxxxx) {25 in(1/2+122/x Cood+x seed-1)-2! (1+cood)]} (14) + Dim (K Knee a) {2 ClK(\12+12-2/xcord+xned-x)-Cl'K[Xo(seed+1)] -Cik[R(seed+1)-2R(1+Cood)]} + Coo(x g cood) {Six(1/x2+1/2-28 cood+ x cood-1)-six[x0(1+cood)]} -sin(Kxcod){CiK(x2+12-2/coox+2cox+2cox-2)-cik[x6(1+ coxd)]} -Cosk(xcosk+2lsind) (Sixfx-1)cood) (+cood) -Six(x2+1-2)cood+xcood-1cood) + sink(x802a+2 laina)[Cit][x-2 (coa)(1+cod)]-Cik(V/2+/2=2/cod+/2cod)]



APPENDIX E

Given I= SS = xp[-]K(x+x+r,+)] 1 dxdx, (1)12,4= V(X,+X,cosd+X,cos2d)2+(x,sind+x,sin2d)2 Let m/t= x,+x,cosd+x,cos2d+1,4, m,=x,sind+x,sin2d mit=xcosd+xcos2d+ro4, r=V(x+xcosd)2+(x sind)2  $m, t = 1 + \chi \cos \Delta + \chi \cos 2\Delta + \Lambda_{24}$ ,  $\Lambda_{24} = \Lambda (\chi + \chi \cos \Delta + \chi \cos 2\Delta)^{2} + (\chi \sin \Delta + \chi \sin 2\Delta)^{2}$ m, t, ] = 70 (H cosd), Ro = 12 = /x2+12+28x602d m,t, ]= x, cond+leosad+120x, /2= 1222+x2+42xcord+212cor2d  $m_1 t_2 = 2 + x_{cosd} + h_{eo}$  =  $x_0 + 2 + 2 \cos \alpha$   $m_1 t_2 = 2 + x_{cosd} + h_{eo} = (x_0 + 2 + 2 \cos \alpha)(1 + \cos \alpha)$ 74+ x,+12,4= m, 2-x, cos x+2x, sin 2x dx= 14 dt m,t, 2(m,t) = x+xcosd+rcosad

- 2(m,t) = x+xcosd+rcosad

- 2(m,t) - 1004[20cosd+xcos2d+ro4] mit 2 (m,t2) = 24+ 20 cod + (l+124) co2+

124 [24] [24] [24]

124 [24] [24]

124 [24] and

Is = expected should share end the text of the state (2)=- eikxcusa [-j2kxsinia {cikmt-cikmt-j(sikmt-sikmt)}] + Expired Clickxaina (eikmita 2/mita) eikmita (mita) d k4 (3)

2 cord - 2x 2 in 2 - 2 - 2 cord - 24 cord - 124 - (2+ 124) 70 cond-2×4 2in2d-70 cond-×4 con2d-104=-(x4+104) (4) - Socik(x4+1204) x4+ cord+ 2002 2d dx4



Let = - Sh-jk(x+1204) x+xcorx+ 1204 cor2d dx4

A32 - Se - 1204 (x-cord+x-cor2d+204) (5)  $m_2y = r_04 + (r_4 + r_0 \cos d)$ ,  $m_2 = r_0 \sin d$ m27 = R + x cost = xo (1+cost) = m, t, ]o rough dy  $m_2 y = R_0 + 1 + \chi \cos d = m_1 t_2 \int_0^{\infty} dx_4 = \chi_4 + \kappa_0 d = m_2 y - \chi_0 \cos d$   $m_2 \frac{1}{2} = \frac{(\chi_0 \sin d)^2}{(\chi_0 + \chi_4 + \chi_0 \cos d)} = \chi_0 - \chi_0 - \chi_0 - \chi_0 \cos d$ 164 = m2 (22+1), x4+x602d = m2 (42-1) A32 = - exxers of = ixm y [m2 (y2-1)+ m2 (y2+1)cos2d] dy

m2 (y2+1)+ m2 (y2-1) cos2d+2xcod mid

sixx cosm (32 1) = 2 (y2+1)+ m2 (y2-1) cos2d+2xcod mid =- e xxx cord 5/2-1/4m, y(= 1) dy (6) Lety=Km2(4 tana)=Km24+Kx sinta, -jKmy=ju+jKx sind cord 21= Km24+KX sind = KX0(1+ secd), 4= K/20+2+x secd) 2-Km2y, 2= KX. (1+co2d), 2= K(20+8+Xco2d)  $A = -2e^{iKx} \cdot 2ecd \int_{K(\Gamma_{0}+l+x,2ecd)} \int_{Kx(l+cood)} \frac{1}{kx} \frac{1}{k$ m27=174+x+xcusa+lcusad, m= xsina+lsin2d my = 170+ 1 cos2d+ x cosd = m, t, ], my = 170+2 cos2d = (x+2 lcosd)(1+cosd) = m, t, ], 2xcoodsin2d+lsin22d = 12 = m2 (42+1)  $\begin{array}{c} \chi_{4} = \frac{1}{2y}(y^{2}) \\ \chi_{4} + \chi_{600} d + \chi_{600} d = \frac{m_{2}}{2y}(y^{2}) \\ \lambda_{3} = e^{i\pi(\chi_{600} d - 2\log^{2}d)} \int_{y_{1}}^{y_{2}} \frac{1}{2y}(y^{2} + 1) + \frac{m_{2}}{2y}(y^{2} + 1) \cos 2d dy \\ \chi_{1} = \frac{1}{2y}(y^{2} + 1) + \frac{m_{2}}{2y}(y^{2} + 1) \cos 2d + m_{1}\sin 2d \end{array}$ / gaind and (xo sind + lsin 2d) = main 2d





```
30
Given I4= SISREJK(X4+X-17,4) dX4dX,
                       APPENDIX F
  12,4=1(x,+x,cosd+x,cos2d)2+(x,sind+x,sin2d)2
 m,t,]= 26(1-cod), m,t,]=12/xand-scored, 12= x, 10= x, 10= x2+172/x and
                                               Tel=X+2/and
 m, +2] = 1/20-1-x cust
  m_1 t_2 J_{\beta} = \pi_{R} - x_0 \cos \lambda - 2 \cos^2 \lambda = (x_0 + 2 \cos \lambda)(1 - \cos \lambda)
  702 A22
my, = x(1-cosd), m2 y= 100-l-x0exxd
   re=-K(rel-xseca), 2, =-Kx(1-seca)
   702 A43
   m24= 12 x cond-leased, my=(x+2/cond)(1-cond)
    2= K(x +2 (and - x Cond - 2 (cond - 2 (sin d)
        = K[x (seed-1)+2 & (+ cood)]
     u = K(X_0 \operatorname{sec} + 1 - R_{01})
   -sksin2d[=
    eikxsoeds2cik(xsoed+l-rol)-cik[x(soed-1)]
                        - CiKIX (seed-1)+2/(1-cood)]
         +j'[25ik(xseed+l-rol)-Sik[xo(reed-1)]
                      - SiK[X0(aced-1)+21(1-cood)]]}
    - eikxand {cik(r-xood-l)-cik[xo(1-and)]
            -J'[Sik(ro-xcosd-R)-Sik[xo(1-cood)]]}
     + EJK (18co2d-2/sinid) { Cik/(x+2/co2d) (-co2d) + Cik/(2-xco2d-1602d)
           -j[Sik[(x+2lcod)(1-cod)]-Sik[rtkcod-lcos2d]]}
```



-sk sind I = Coz(KX, secd) {2CiK(xsecd+l-1xo2+l2+2lxoCozd)-CiK[xo(secd-1)] -Cik[x,(seed-1)+21(1-cood)]} +sin(Kxsecd){25ik(xsecd+l-Vx2+l2+2lxcod)-sik[xolsecd-V] - Sik[xo(sed-1)+28(1-co2d)]} - cos(Kx, cosd) {cik(Vx2+12+21x, cosd-xcosd-x)-cik[x, (1-cosd)]} + sin(Kx cood) {sik(1/x2+1/2+2/xcood-x, cood-1)-sik[x, (1-cood)]} + Colkx cod-2laind) {Cik[(x+2kood)(1-cood)] - Cik( 1x2+12+2/x, cood-x cood-lased)} - sink(xcod-2lsin2) [5 [K[(x+2lcood)(1-cood)] +i/cox(xsec){25ix(xsed+l-1/x2+l2+21x,cod)-5ix[x(secd-1)] - Sit[xo(seed-1)+2l(1-cosd)]} -sin(Kxsecd){2CiK(tgecd+l-1x2+l2+2lxcod)-C(t[xsecd-1)] - Cit[xo(seed-1)+2l(1-coad)]} + col(xxcos) {Sik(/x2+12+21xcosd-xcosd-1)-Sik[x(1-cost)]} + sin(kx cood) {Cik(/x2+12+21x cood-x cood-x)-cik[x, (1-cood)]} - Co2K(xcod-2/sin2){SiK[(x,+2/cod)(1-cod)] -SiK(\x2+/2+2/xcod-xcord-/co2d)} - Ain K (x cozd-2 faind) {Cik[(x+2 (cood)(1-cood)] -Cik(VX2+12+21xcood-xcood-1coo2d)}



Given  $I_5 = \int_0^1 \int_0^1 e^{5K(X_3 - X_1 - X_{13})} dx_3 dx_1$   $K_{13} = \sqrt{(X_1 + X_0 \cos 2d - X_3 \cos 2d)^2 + (X_0 \sin d - X_3 \sin 2d)^2}$ 32  $m_1t = R_{13} + R_1 + R_0 \cos \alpha - R_3 \cos 2\alpha$ ,  $m_1 = R_0 \sin \alpha - R_0 \sin 2\alpha$   $m_1t = R_{03} + R_0 \cos \alpha - R_0 \cos 2\alpha$ ,  $R_0 = \sqrt{R_0^2 + R_0^2} - 2R_0 R_0 \cos \alpha$ mit= 1/3+l+x cord-7, corad =1/(x3-x6co2d)2+(xsind)2 , 12=1(x=x5000d-1coo2d)2+(x sind+lain2d)2 m, t, ] = 2 (1+ cord) M, t, ]= 10 + x cord- 2cord , 1200 = x0, 12 = N2 + x2 - 2lx cord m, t\_2]=140+8+x0cosd, 120=182+x02+28x0cosd m, t2] = 12+ x Cord+2 laind, 12= 122+(2 laind)=  $m_1 dt = [x_1 + x_0 - x_3 \cos \alpha + 1] dx_1 = \frac{m_1 t}{r_{13}} dx_1, dx_1 = \frac{n_1 s}{t} dt$   $\frac{\partial (m_1 t)}{\partial x_3} = \frac{x_3 - x_0 \cos \alpha - x_1 \cos \alpha t}{r_{13}} \cos \alpha t = \frac{x_3 - x_0 \cos \alpha - (r_1 + x_1) \cos \alpha t}{r_{13}}$  $\frac{1}{m_1 t_2} \frac{\partial (m_1 t_2)}{\partial x_3} = \frac{x_3 - x_0 \cos d - (n_{13} + l) \cos 2d}{n_{13} (n_{13} + l + x_0 \cos d - x_3 \cos 2d)}$   $\frac{1}{m_1 t_1} \frac{\partial (m_1 t_1)}{\partial x_3} = \frac{x_3 - x_0 \cos d - n_{03} \cos 2d}{n_{03} + x_0 \cos d - x_3 \cos 2d}$ m, = 3(m, L2)  $T_{5} = e^{j}K_{\lambda}co_{\lambda}d\int_{0}^{\lambda}e^{j}K_{\lambda}x_{3}\sin^{2}d\int_{0}^{\lambda}K_{\lambda}$ = esktocood [esakzainas [cikm,t-cim,t-j(sikm,t-sikm,t)]  $= \underbrace{e^{jk} x_{cosd}}_{j2k} \underbrace{e^{jk} x_{sind}}_{j2k} \underbrace{e^{-jk} x_{m_1 t_2}}_{j2k} \underbrace{e^{jk} x_{m_1 t_1}}_{j2k} \underbrace{e^{jk} x_{m_1 t_1}}_{j2k} \underbrace{e^{jk} x_{m_1 t_2}}_{j2k} \underbrace{e$ K/6 cord + K2×3 sind-KM, t,= K(x3-1203) KX cod+ K2 X 2 min 2 - Km, t2 = K(X3-l-123)



 $R_{5} = -\int_{0}^{R} e^{jk(x_{3}-l-n_{2})} [x_{3}-x_{3}\cos d - (n_{2}+l)\cos d] dx_{3}$   $\frac{n_{2}}{n_{2}} [n_{2}+l+x_{3}\cos d - x_{3}\cos 2d] dx_{3}$  $+ \int_{0}^{R} e^{jk(x_{3}-R_{03})} \left[ x_{3}-x_{0}\cos d - R_{03}\cos 2d \right] dx_{3}$   $+ \int_{0}^{R} e^{jk(x_{3}-R_{03})} \left[ x_{3}-x_{0}\cos d - x_{3}\cos 2d \right] dx_{3}$   $+ \int_{0}^{R} e^{jk(x_{3}-R_{03})} \left[ x_{3}-x_{0}\cos d - x_{0}\cos 2d \right] dx_{3}$ Mos [ Mos + X , Cood - 1/3 coo 2d] 1203 = 1(x3-x6cord)2+(xosind)2 my=10-73+70 cood, m= xosind, x-12=-my+70 cood m24= 102-8+xocosa, m24= x0(1+cosd) = m, t, ]0  $m_2 dy = \left[\frac{x_3 - x_0 \cos d}{r_0 \cos d} - 1\right] dx_3 = -\frac{m_2 y}{r_0 \cos d} dx_3, dx_3 = -\frac{r_0 \cos dy}{y}$   $\frac{m_2}{y} = \frac{(x \sin d)^2}{r_0 - x_1 + x_0 \cos d} = r_0 + x_3 - x_0 \cos d, \quad x_0 \cos d(1 - \cos 2d) = m_1 \sin 2d$ 21= Km2 (y+tend)= Km2y+K20 sind cood -jkm2y=-ju+jkk sin2d 22= K(120-8+X0seed), 21,= KX0(seed+1) u= Km2y, u=K(12,-2+xcosh), u=KX,(1+cosh) Asz=2eikkonedScik(nopl+xoned)-Cik[xo(neex+1)] -J[Sit(nor-l+xored)-Sit[xo(seed+1)]]} - Exxcosdscik(notex cosd)-cik[xo(1+cosd)] - I[Sik(ro=l+xocod)-Sik[xo(1+cod)]}



34  $A_{53} = -\int_{0}^{\rho} e^{j\kappa(x_{3}-\lambda-n_{83})} [x_{3}-x_{5}\cos \lambda-(n_{3}+l)\cos 2\lambda]} dx_{3}$   $\frac{n_{83}}{n_{83}} [n_{83}+l+x_{5}\cos \lambda-x_{5}\cos 2\lambda]} dx_{3}$  $R_{3} = \sqrt{(x_3 - x_0 \cos t - l \cos 2d)^2 + (x_0 \sin d + l \sin 2d)^2}$ my= Re3-x3+x cosd+lcos2+, m= xaind+lsin2d 73-1-123 = -m2y+xewd-2/sin2d, 12=1/2+x2+2/xce02/ m24=128+76 Cord -28 sin2d , Pel=120 + (2/sind)2 Mzy= Rgo + rocusd+ 1 cossd m2 dy = [ x3-x0co2d-lcos2d-1]dx=-m2y dx3, dx=- 12 dy  $\frac{m_2}{3} = \frac{(\chi_0 \sin d + 2 \sin 2 d)^2}{(\chi_0 - \chi_0 + \chi_0 \cos d + 2 \cos 2 d)}$   $\chi_3 = \frac{m_2}{3} + \chi_0 \cos d + 2 \cos 2 d$   $\chi_3 - \chi_0 \cos 2 d - 2 \cos 2 d = -\frac{m_2}{2} (y^2 - 1)$ = 123+13-16COX-1COXIX 1+7, cust (1-cus2x)-1202d= 2x cood sind + l sin 2d = 1 Ain 2d u= 12+x secd, u,= 120+x secd+l 2 = Km2y, 2= K(12+x cord-2(sin2), 2= 12+x cord+1 cor2d A53=-2= Six 8 seed { Cit(1/20+x 20ed) - Cit(1/2+x 20ed+1) + esk(xcod-2lsind){Cik(ny+xcod-2lsind)-Cik(ny+xcod+lcod) - J[Six(120+xcood-2lsind)-Six(2+xcood+levo2d)]}



j2ksind I= 2e kx seed feck (ro-l+x seed) + Cik(ro+l+x seed) - Cik[x becd+1)z-cik(/ze+x seed)-j[sik(/z-l+x seed)+sik(/z+l+x seed) -SiK[x(ora+1)]-Sik(ze+xora)]]-eikxorafcik(ze-l+xora) + Cik(12+2+xcood)-2cik[x,(1+cood)]-i[sik(12-2+xcood) +Sik(12+2+xcood)-2sik[x,(1+cood)]]+eik(xcood+2loind) [Cik(Tge+x cood+2lain)-cik(12+x cood-leased)-)[sik(7x+xcood+2lain)) -Sik(rot+xcood-leased)]]+eik(xocood-slainid)fik(rot+xcood-slainid) -Cik(rot+xcood+leased)-J'[Sik(rot+xcood-slainid)-sik(rot+xcood+leased)]] 12Ksin2 2]= 2coo(KKoseed) [CiK(V/2+x2-2/18, cood-l+ 2, seed) + Cik(N2+x2+21x, cosd+1+ 25 seed)-cik[x(seed+1)]-cik(Nx2+(2/2mid)2+x2ed)} +2 sint & seed) [Sik(N/2+x2=2/x6 and -2+x seed) +Sik(N/2+x2+2/x6 and +2+x seed) - Sik[1/0(seca+1)]-Sik(1/20+(2Prina)2+20seca)} - Cos (Kx cod) {cik( 1/2+x221x cod - 2+x cod+)+ cik( 1/2+x2+21x cod+)+xcod)-2(ik)x(1+cod)} - Din(KXGODA)SiK(VP+X=2/xcood-++XGODd)+SiK(VP2+X2+2/xcood+++xcood)-25iK[X6(1+cood)] + Cost(x cosat 2 lain d) for K(K3+(2 lain a) 2+x cosat 2 lain a) - Cit (12+x32k and +x cosat - Ros 2d) + Dink(xcood+2lsind){SiK/x3+(2Rsind)2+xcod+2lsind)-SiK(1824x22) Read + cood-Pcoo2d)} + Cook(xcood-2/sind) [ ( K(VX 2+(2/aind)2+) Cood+2/sind)-( K(V2+x2+2/xcood+xcood+) Cood+2/cood+2/sind) - ( K(V2+x2+2/xcood+xcood+xcood+2/cood)) + Link(x602d-2/sind) SK(V2+x2+x2/xcood+xcood+xcood+xcood+xcood+xcood+xcood+xcood+xcood+xcood+xcood+xcood+xcood)} +1-2col(x ever){Six(V/2+x=2/xcord-l+x,red)+Six(V/2+x2+2/xcord+l+x,red) -Sik[x(seed+1)] +Sik(NX62+(2lsind)2+X0 seed)} +2sin(KX,20e d){C(KW/2+x2-2/x,000d-2+x,000d)+CiK(V/2+x2+2/x,000d+2+x,000d) + Coo(KXCood) {Sik(1/2+1x=2/xcood-1+xcood)+Sik(1/2+1x=2/xcood+1+xcood) -25:K[x6(1+cod)]}-sin(Kx6cod){Cik(N2+X2-21x6cod-2+xcod)}
+cik(N2+X2+21x6cod+1+x6cod)-2cik[x6(1+cod)]} - Cozk(xond+2laind) {Six/x3+(2laind)2+x cond+2laind)-Six(x2+x22/x cond+xcond-xcond)} +Mink(xcond+2laind) {Cik(xx2+(2laind)2+x cond+2laind)-Cir(x2+x22/x cond+xcond-xcond)} -Cosk(xcood-28sind)(Sik(/x2+(28and)2+xcood-2laind)-Sik(/2+x+1/xcood+xcood+levered)) 1 sink (x50d-2/sind) {cik(/x2+(280ind)2+1>602d-2/sin2d)-Cik(/2+1x2+2/2/cod+xcord+1co2d)}



 $I_{7} = \int_{0}^{R} \int_{0}^{1} e^{-\lambda k(x_{2}-x_{1}-R_{12})} dx_{2} dx_{1}, r_{12} = \sqrt{(x_{1}-x_{0}-x_{2}cos2d)^{2}+(x_{3}sind+x_{3}sin2d)^{2}}$ m, t = 1,2+x,-xcesd-xcos2d, m, = 2, sind + 22 sin 2 d 72-x,-r,2=-m,t-x,cosd+2x,sin2d, Ro=1x2+x2+2xx,cosd m, = 12+1-xcosd-x2cos2d = 1(x2+x6co2d)2+(x,2ind)2 12= V(x+x0cord-fcor2d)2+(gaind-Asin2d)2 Mit= no= x cosd-x cos2d M, t] = rog x, cost-lossed 1208= N82+x2+21x0cord m, 2]=Re-Xoesd+2lsin2d 120=N82+x3=28xevsd miti]= 20(1-cosd) rel=12+(2/sind)2  $m_1t_2$  =  $n_0$  -  $\pi$  easy +1m, dt = [x-xevrd-xevr2d+1]= m,tdx, , dx = " dt  $\frac{\partial(m_1t)}{\partial X_2} = \frac{\chi_2 + \chi_0 \cos 2d - \chi_1 \cos 2d}{\chi_1 \cos 2d} = \frac{\chi_2 + \chi_0 \cos 2d - (\chi_1 + \chi_1 \cos 2d)}{\chi_1 \cos 2d}$  $\frac{1}{\sqrt{3}} \frac{\partial (m_1 t_2)}{\partial x_2} = \frac{\chi_2 + \chi_0 \cos \alpha - (\sqrt{\chi_2} + \ell) \cos \alpha}{\sqrt{\chi_2 (\sqrt{\chi_2} + \ell) - \chi_0 \cos \alpha - \chi_0 \cos \alpha}}$ 7/22(1/2+1-xcord-x2cor 2d)  $\frac{1}{m_i t_i} \frac{\partial (m_i t_i)}{\partial \chi_2} = \frac{\chi_2 + \chi_0}{r_{o2}} \frac{\cos \alpha - r_{o2} \cos 2\alpha}{r_{o2} (r_c - \chi_0 \cos \alpha - \chi_0 \cos 2\alpha)}$ In = eikxcord fl szkxzsinzd skm,tz ejt dt = eskxocood [soxxsin2] [cikm,t,-isikm,t,-sikm,t,-sikm,t,)}  $-\frac{e^{jk}\kappa_{co2d}}{s^{2}\kappa_{co2d}}\int_{0}^{k}\frac{1}{2}\kappa_{x}s_{co2d}\frac{e^{jkm_{i}t_{2}}}{s^{2}\kappa_{i}t_{2}}\frac{e^{jkm_{i}t_{2}}}{3\kappa_{2}}-\frac{e^{-jkm_{i}t_{i}}}{m_{i}t_{i}}\frac{J(m_{i}t_{i})}{3\kappa_{2}}d\chi_{2}$ An = - Exkound of 2xx sind (eixmit) /mit) of /mit) of dx2 -KKCOOX+2KK2sin2x-KM, E2=K(K2-l-122) - KX, Cos +2KX, sin2 - Km, t, = K(x-102) An= - Seitke-l-Res/1x2+ x0 cosd-(rp2+1) cos2d] dx2 + Soeik(x-roz) [x2+20cosd-rozcos2d] dkz



 $A_{72} = \int_{0}^{8} e^{j\kappa(x_{2}-x_{02})} (x_{2}+x_{0}\cos x - x_{0}\cos 2x) dx, \quad R_{02} = k(x+x_{0}\cos x)^{2} + (x_{0}\sin x)^{2}$ m2y=102-72-2000d, 2-102=-m2y-xcood, m2= 20 sind  $m_2 y_2 = R_0 - \chi_0 \cos d - l$ ,  $m_2 y_1 = \chi_0 (1 - \cos d) = m_1 t_1 J_0$   $m_2 dy = [\frac{\chi_2 + \chi_0 \cos d}{R_0 z} - 1] d\chi_2 = -\frac{m_2 y}{R_0 z} d\chi_2$ ,  $d\chi_2 = -\frac{R_0 z}{y} dy$  $m_2/y = \frac{(\chi_0 \text{ sind})^2}{\chi_0 - \chi_0 - \chi_0} = \frac{(\chi_0 \text{ sind})^2}{\chi_0 - \chi_0 - \chi_0} = \frac{\chi_0 + \chi_0 - \chi_0}{\chi_0 - \chi_0} = \frac{\chi_0}{\chi_0} (y^2 + 1)$ 2+x, cood=-m2(221), -7, (1-co22x) cood=-2x, cinderd=-main 2d -u=km2(y-tand)=kmy-Kx sind -jkmy=ju-jkt sind  $-\mathcal{U}_{2}=K(n-x\operatorname{secd-l}), +\mathcal{U}_{1}=-k\times_{0}(1-\operatorname{secd})$ 2= Km=y, 2=12-16 cosd-2, 21=26 (1-cosd) A72=20-1/x secd {Cik(xsecd+l-rol)-Cik[xo(seed-1)] +1)[six(x, seed+2-1702)-six[x, (see x-1)]]} - Ejkx cod { (1/2 - 2 cod - 1) - (i k[x (1-cod)] ->[Sik(rg-xcord-R)-Sik[x,(1-cord)]]} A73-52 e3k(2=1-12)[x2+x,co2d-(12+1)co22d] dx2

122 [122+1-xc02d-(12+1)co22d] 12= 1(x2+x002d-8co2d)2+(xsind-8sin2d)2 my=12-x-76cord+8cos2x, m= Ksind-lsin2x 72-1-1/2= -x cood-2/sind-May, 12=1/2+x2-2/x cood , 12= 1/2+(2/sind)2 my= 12- 2, cos d-2/ sind m2y= 120-x cod+ 1co22d



 $\frac{m_2}{y} = \frac{(x_0 \sin \alpha - l \sin 2\alpha)^2}{r_0 - r_2 - r_0 \cos \alpha + l \cos 2\alpha}$ = 12+x+x0cord-1cor2d  $\frac{R_{2}}{2y}(y^{2}+1), (x_{0}\cos d - f\cos 2d)\cos 2d + f - x_{0}\cos d = R_{0}\sin^{2}2d - 2x_{0}\cos 2d \sin^{2}d = -m_{0}\sin^{2}2d - 2x_{0}\cos 2d \sin^{2}d = -m_{0}\sin^{2}2d$   $\frac{R_{2}}{2y}(y^{2}+1), (x_{0}\cos d - f\cos 2d)\cos 2d + f - x_{0}\cos d = -m_{0}\sin^{2}2d - 2x_{0}\cos 2d \sin^{2}d = -m_{0}\sin^{2}2d$   $\frac{R_{2}}{2y}(x_{0}\cos d - f\cos 2d) = -m_{0}\sin^{2}2d - 2x_{0}\cos 2d \sin^{2}d = -m_{0}\sin^{2}2d$   $\frac{R_{2}}{2y}(x_{0}\cos d - f\cos 2d) = -m_{0}\sin^{2}2d - 2x_{0}\cos 2d \sin^{2}d = -m_{0}\sin^{2}2d$ A73 = - E'K(xcosa+2/eind) [1/2 = ikm2 y = m2/y = 1) + (m2/y = 1) tos20] dy

[ m2/(y2+1)+ m2/(y2-1) cos20 - m2 sin 2 d 2=-Km2(y-tand)=-K(m2y-x sin2d+2/sin2d)
-Km2y=2-K(x sin2d-2/sin2d) 2=15(x0sex-reg), 2= K(x0secd-rego-1) 21 = - Kmy, 21= 12= 25 Cood-28 sind, 21=16-25 Cood+ less 2d Az=-2ejk/socofcik(xocod-/ze)-Cik(xocod-l-ngo) +5/5:K(x seed-12)-Sik(x see 2-1-120)]}
+Eik(x cood+2/sin2d)Cik(12, x cood-2/sin2d)-Cik(12-x cood+)cood)
i[e: 1/17-7 - J[Sik(M-xcood-2lsin2d)-Sik(M-xcood+lcoo2d)]} J2KSin 2 [ = 2 EJKX o seed SCIK(xseed+l-12)+CiK(xseed-l-120) -Cikix (see x-1)]-cik(xsee a-rg)+j[sik(xsee x+l-rg) + Six (xose x-1-120)-5ix[xo(seed-1)]-Six(xosex-120)]5 - E-1/KX0cord [Cik(re-l-xend)+Cik(re+l-xend)-2cik[xo(1-cord)] ]
- i[Sik(re-l-xend)+Sik(re+l-xend)-2sik[xo(1-cord)]] + (-1/K(xcod+2/2in2d) {(1/K(12-xcood-2/sin2d)-cik(12-xcood+2coo2d) -1[Sik(12-xco2x-28sin2x)-Sik(120-x,co2x+2co2x)]} + E-SK(xcoad-2lain2d) {C(K(7g-xcood+2lain2d)-Ciklig-xcoad-2coa2d -1[Sik(Rg-x,cord+2fsin2d)-Sik(Rofx cod-love 24)]}



J2KsinZI= 2 coz(Kx secd) {Cik(x secd+1-1/12+x2+21x602d) + CiK(x seed-2-1/2+x2+28x602d)-CiK[x6(ved-1)]-CiK(xseed-1/2+(2/aind)2)} +2 sin(KX,secd){Sik(X,secd+l-V2+X,2+2lx,cesd)-Bik[xo(seed-U] +Six (xorecd-l-N/2+X2=2/x cord)-Six(xored-Nx2+(2/sind)2) - Cos (KK, Cosd) (Cik(1/2+X2+2/xcosd-1-x, cosd)+(1/4/1/2+x2-2/xcosd+1-x cosd) +sin (KKCvzd) {S(K(VR2+X2+21xcvsd-x-xcvzd)+SiK(VR+X=21xcvzd+x-xcvzd)}
-25iK[xo(1-cvsd)]} + cost(xcosd+2/sind) {Cik(/x2+(2/sind)2-xcosd-2/sind)-Cik(/x2+2/xcosxcosd+/cos2d)} - sinklycood+2laind) (Siklyx2+(2laind)2-x cood 2laind)-Siklikx-2lycod-xcood+leus 2d) + cosk(xood-2/aird) [ik(Vx7(2/aird) = x cosd+2/aird)-cik(V27x3-2/xcosd-x cosd+2cos2d)]
- Dink (xcosd-2/aird) [Sik(Vx7(2/aird)= x cosd+2/aird)-Sik(V27x2+2/xcosd-x cosd-2cos2d)] +1[2001/Kgsecd){SiM(x 20cd+2-1/84x2+2/x cood-Six[x 6/20cd-1)} +Six(xosecd-8-482+82-21xcood)-Six(xoced-1x2+(2laind)2)} -2 sin(kx, seed) { (x, x, seed+1-1/2+x, +2/x, sod) - cix[x, (seed-1)] +CiK(x,secd-l-1/2+x,2-2/x,cosa)-cik(x,secd-1/x,2/2/sind)2) + Cos/Kxcod) ( 61/12+x2+2/x cosd-l-xcosd) - 251/x[x(1-cosd)] + Sik(V/2+x02+21x0cosd+1-x0cosd)} - sin(Kxcood){cik(V2+x02+21xcood-1-x0cood)-2(11x[x0(1-cood)] + Cik(N12+x2+21x0 cusd+1-x0 cusd) } -Cook(xcood+2lsind){Sik(x2+6lsind)=xcood-2lsind)-Sik(x2+x2-2lxcood+lcood)} -Ain K(xcord+2laind) {cik(/x2 (2laina)2-x600d-2 Raina)-cik(/l2x2-2lxcord xcord+lcor2d)} - Cost(Kood-2/sind) Git/(x2(2/sind) = x cood+2/sind) - Sik(1/7-x+2/tsood-xood-xcood-lcoozed) } -sink(xcosa-2laina){cik(xx+6loina)2xcosa+2laina)-cik(xx+2/xxcosa-x



For 70 = 2 levent + b, beck &

22+4 seed-N412co2+4lbco2+4l2sin2d= 21+breed-N412+4lbco2+12=

 $-2l+b \sec \lambda -2l-4lb \cos \lambda + b^2 \sin^2 \lambda = b(\sec \lambda - \cos \lambda)$   $= b \sin^2 \lambda$ 

1+1-cosd + 62 sin2d - 2 | cosd - 1 cosd + 2 | cos2d - 1 = 6 sin2d





